Matrix-geometric stationary distribution for the PH/PH/1R queue

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ABSTRACT

This paper is devoted to the study of the PH/PH/1/r queueing systems. Using matrix calculus together with the stochastic interpretation of distributions of PH-type, we obtain the steady-state probabilities in a matrix-geometric form. Numerical examples are also given.

RESUME

Cet article analyse la file d'attente PH/PH/1/r. En utilisant le calcul matriciel ainsi que l'interprétation probabiliste de la distribution de type PH, nous obtenons les probabilités stationnaires sous forme "géométrique matricielle". Des exemples numériques sont présentés.

N.B.

Ce travail a été terminé pendant la visite du professeur P.P. BOCHAROV à l'INRIA du 9/3/84 au 9/4/84, dans le projet "Modélisation et évaluation des systèmes informatiques" - responsable scientifique G. FAYOLLE.

1 - INTRODUCTION

A single server queueing system (QS) with a buffer of finite capacity r ($1 \leq r < \infty$) is studied . The customers arrive according to a renewal process with probability distribution function (P.D.F.) $A(x)$. The service times of customers are i.i.d. random variables with P.D.F. $B(x)$. An arriving customer finding a full buffer is lost. Using Kendall's notation, this QS is denoted GI/GI/1/r.

A huge literature has been devoted to special cases of this QS. In the present paper we shall consider a situation, where steady-state distribution is given in a matrix-geometric form [1-6]. In the past, such a form has been obtained by Basharin [1,2] for M/HM/1/r $\leq \infty$ (HM $\equiv$ hyperexponential). The system M/HK/1/r has been analysed in [4] (K $\equiv$ coxian, i.e. rational Laplace transform). The matrix-geometric solution for steady-state probabilities in the QSs M/PH/1/r and PH/M/c/r was given by Neuts [5]. The symbols "PH" denote a distribution of phase type [7]. Recent results have been obtained in [6] for the QS HM/HM/1/r. They rely on some arguments developped in [3], where the arrival process is considered as one server QS of type HM.

In the present paper we analyse the system PH/PH/1/r. The main result tells that the steady-state probabilities have a matrix-geometric form. In section 2 we briefly mention the notations and some properties of the PH-distribution, which are crucial in the proof of the main theorem of section 3. Numerical examples are also given.

2 - PH-distribution

A P.D.F. $F(x)$ of a non-negative random variable is called a PH-distribution [7] if it can be written as

$$F(x) = 1 - \bar{f}^T e^{Gx}\, \bar{1}, \quad x \geq 0, \tag{1}$$

where:

- the m-dimensional row vector $\overline{f}^T$ satisfies

$$\sum_{j=1}^{m} f_j \leqq 1, \quad f_j \geqq 0, \quad j = \overline{1,m} \ ;$$

- G is an (mxm)-matrix such that

$$\sum_{j=1}^{m} G_{ij} \leqq 0, \quad G_{ij} \geqq 0, \quad G_{ii} < 0, \quad i,j = \overline{1,m} \ ;$$

- $\overline{1}$ denotes the column vector having all its components equal to one.

The pair $(\overline{f},G)$ is then called the PH-representation of order m of the P.D.F. F(x).

A P.D.F. F(x) of the PH-type has a probabilistic interpretation which we show right now.

Let $\nu_1,\dots,\nu_m$ be real numbers,

$$\nu_i \geqq -G_{ii}, \quad i = \overline{1,m} .$$

Define θ_{ij}, $i,j = \overline{1,m}$,

$$\theta_{ij} = \begin{cases} 1 + \dfrac{G_{ii}}{\nu_i}, & i = j, \\ \dfrac{G_{ij}}{\nu_i}, & i \neq j. \end{cases}$$

Then

$$\sum_{j=1}^{m} \theta_{ij} \leqq 1, \quad \theta_{ij} \geqq 0, \quad i,j = \overline{1,m} .$$

Consider an open network with m nodes as shown on Figure 1.

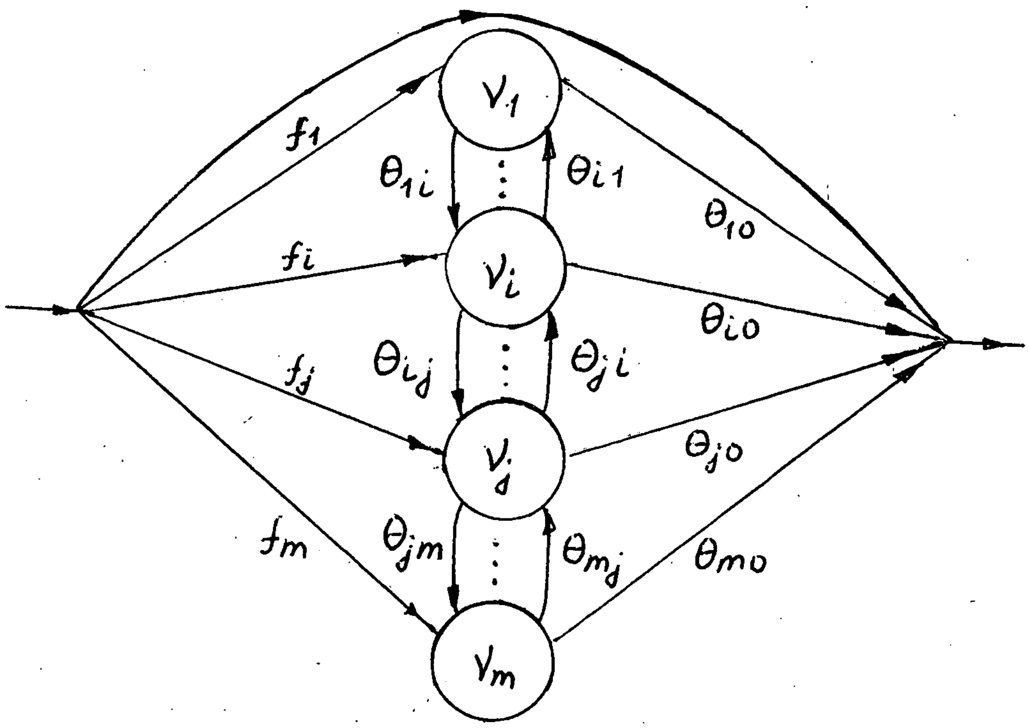

Figure 1

An arriving customer enters node i, $i = \overline{1,m}$, with probability f_i, or is rejected with probability

$$f_o \overset{\text{def}}{=} 1 - \sum_{j=1}^{m} f_j .$$

His sojourn time at node i has an exponential distribution with parameter ν_i. After completing service at node i, this customer enters node j with probability θ_{ij} and with proability

$$\theta_{io} \overset{\text{def}}{=} 1 - \sum_{j=1}^{m} \theta_{ij}$$

he leaves the network. Moreover we assume that there is always at most one single customer in this network and τ will denote his sojourn time. Let $\xi(t)$ be the index of the node visited by this customer at time t. The process $\xi(t)$ defined only for $t \in [0,\tau)$ is an homogeneous terminating Markov process with generator G, where

$$G_{ij} = \begin{cases} \nu_i(\theta_{ii} - 1), & i = j, \\ \nu_i \theta_{ij}, & i \neq j. \end{cases}$$

Then, $F(x)$ is the P.D.F. of the time until the process $\xi(t)$ terminates.

A PH-representation $(\bar{f}, G)$ is called irreducible, if $f_o = 1 - \bar{f}^T \bar{1} \neq 1$ and

$$A = G - \frac{1}{1 - f_o} G\bar{1}\,\bar{f}^T$$

is irreducible.

This case may be interpreted as a random walk of one customer in the closed network shown on Figure 2.

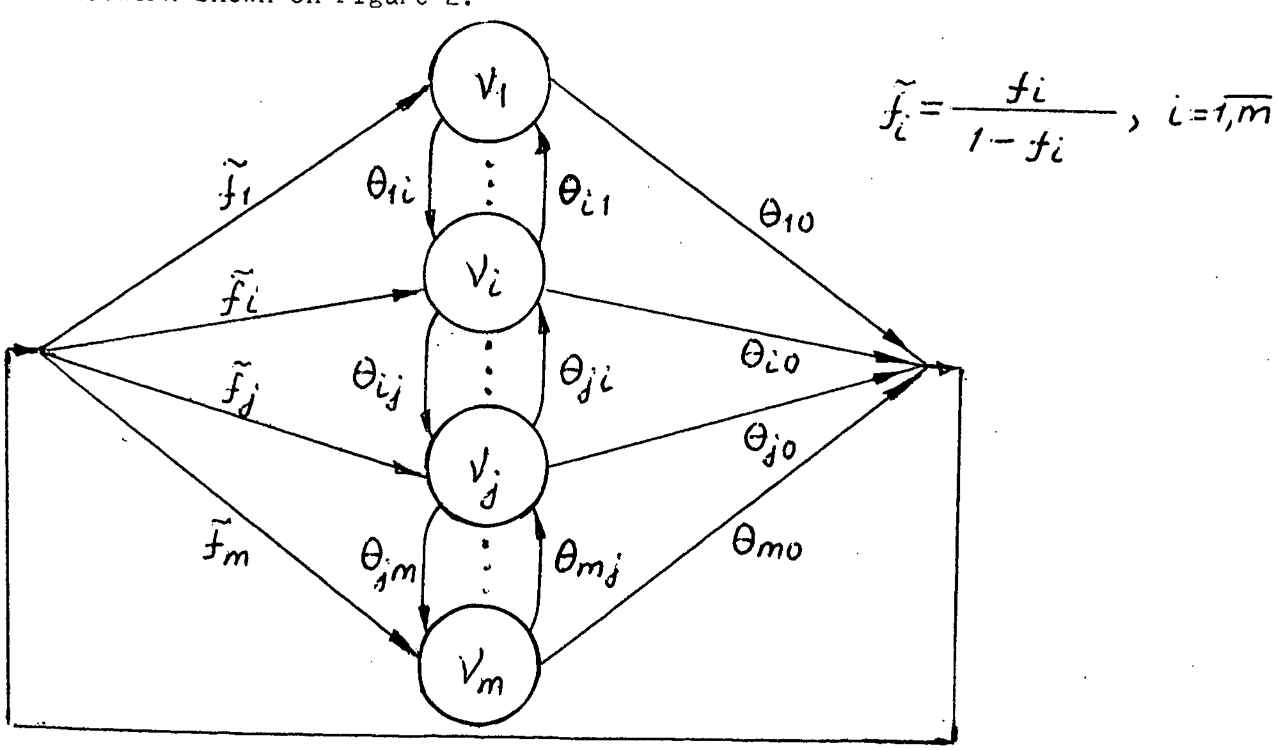

Figure 2

<u>Lemma 1</u>. Let $(\overline{f},G)$ be an irreducible PH-representation. Then the matrix $H = -G$ is an invertible M-matrix.

<u>Proof</u>. It is know [8] that, if a matrix B satisfies the condition

$$B_{ij} \leqq 0, \; i \neq j, \; i,j = \overline{1,m},$$

and if there exists a vector $\overline{x}$ with positive components so that

$$\sum_{j=1}^{m} B_{ij}x_j \geqq 0, \; i = \overline{1,m} \,,$$

then B is a M-matrix. Such a vector for the matrix H is the vector $\overline{x} = \overline{1}$. Furthermore, from [7], we know that the matrix B of an irreducible PH-representation is invertible. This concludes the proof.

$\square$

As the real parts of the eigenvalues of an invertible matrix are positive, we have the

<u>Corollary</u>. Let $(\overline{f},G)$ be an irreducible PH-representation. Then the matrix G is stable.*)

This follows from the fact that a PH-representation is of "exponential type" as introduced in [9].

$\square$

All characteristics of a P.D.F. F(x) with an irreducible PH-representation $(\overline{f},G)$ can be otained from the vector $\overline{f}$ and the matrix G. In particular the Laplace-Stieltjes transforms (LST) of the P.D.F. F(x) is given by

$$F^*(s) = f_o - \overline{f}^T(sI-G)^{-1}G\overline{1} = 1 - s\overline{f}^T(sI-G)^{-1}\overline{1}, \; \text{Re } s \geqq 0. \qquad (2)$$

*) A matrix A is stable if all its eigenvalues λ_i satisfy Re $\lambda_i < 0$. A is substable if Re $\lambda_i \leqq 0$.

The k'th moment is

$$\int_0^\infty x^k dF(x) = (-1)^k k!\ \bar{f}^T G^{-k}\bar{1},\ k = 1,2,\ldots \qquad (3)$$

In the sequal we shall also use the following

Lemma 2. Let $(\bar{f},G)$ be an irreducible PH-representation of order m of the P.D.F. $F(x)$, $\bar{f}^T\bar{1} = 1$. If the (nxn)-matrix H is stable, then the (nmxnm)-matrix

$$\tilde{G} = G \otimes I + (I - \bar{1}\,\bar{f}^T) \otimes H$$

is invertible.*)

Proof. First we shall show that the matrix $G^{-1}(I - \bar{1}\,\bar{f}^T)$ is substable. Therefore let γ be an eigenvalue of that matrix and $\bar{x}$ be the corresponding eigenvector

$$G^{-1}\bar{x} - (\bar{f}^T\bar{x})\, G^{-1}\bar{1} = \gamma\,\bar{x}\,. \qquad (4)$$

Assume that Re $\gamma > 0$. G and G^{-1} being stable, $1/\gamma$ (resp. γ) can not be an eigenvalue of G (resp. G^{-1}). Hence $\bar{f}^T\bar{x} \neq 0$. Furthermore relation (4) is equivalent to

$$\bar{x} = \left(\frac{\bar{f}^T\bar{x}}{\gamma} \right) \left(\frac{1}{\gamma} I - G \right)^{-1} \bar{1}\,.$$

Then it follows that

$$\bar{f}^T\bar{x} = \frac{(\bar{f}^T\bar{x})}{\gamma}\ \bar{f}^T \left(\frac{1}{\gamma} I - G \right)^{-1} \bar{1}. \qquad (5)$$

Using (2) and (5) we get

$$F^*\left(\frac{1}{\gamma} \right) = 1 - \frac{1}{\gamma}\bar{f}^T \left(\frac{1}{\gamma} I - G \right)^{-1} \bar{1} = 0.$$

*) A $\otimes$ B denotes the usual tensor product of the matrices A and B (see for instance [11]).

But the LST of the P.D.F. of a non-negative random variable is strictly positive in the right-hand plane [10]. Hence the equality (6) contradicts the assumption Re $\gamma > 0$ and the matrix $G^{-1}(I-\bar{1}\,\bar{f}^T)$ is therefore substable.

We rewrite now the matrix $\tilde{G}$ as

$$\tilde{G} = (G \otimes I)\ \Psi(I \otimes H),$$

where

$$\Psi = I \otimes H^{-1} + G^{-1}(I - \bar{1}\,\bar{f}^T) \otimes I.$$

Each eigenvalue of Ψ is the sum of the eigenvalues of the matrices H^{-1} and $G^{-1}(I - \bar{1}\,\bar{f}^T)$ [11]. H^{-1} is stable and $G^{-1}(I - \bar{1}\,\bar{f}^T)$ is substable. This implies that Ψ is stable and invertible. Moreover, the matrices $G \otimes I$ and $I \otimes H$ being also invertible, so is $\tilde{G}$ and the proof of Lemma 2 is concluded.

$\square$

3 - QS PH/PH/1/r

Let us assume that P.D.F.s $A(t)$ and $B(t)$ allow the following irreducible PH-representations

$$A(x) = 1 - \bar{\alpha}^T e^{\Lambda x}\,\bar{1},\ x \geq 0,\quad \bar{\alpha}^T\,\bar{1} = 1,$$
$$B(x) = 1 - \bar{\beta}^T e^{Mx}\,\bar{1},\ x \geq 0,\quad \bar{\beta}^T\,\bar{1} = 1,$$

of order ℓ and m respectively.

Using probabilistic interpretation of a PH-distribution, the QS PH/PH/1/r may be considered as an homogeneous Markov process $\{X(t),\ t \geq 0\}$ on the state space

$$X = \bigcup_{k=0}^{r+1} X_k,$$

where

$$X_o = \{(i,0) \mid i = \overline{1,\ell}\},$$

$$X_k = \{(i,k,j) \mid i = \overline{1,\ell},\ j = \overline{1,m}\},\ k = \overline{1,r+1}\ .$$

The irreducibility of the PH-representations of A(x) and B(x) entails that the process X(t) is irreducible. We denote the stationary probability of the state $x \in X$ by p_x. Define the vectors

$$\overline{p_o} = (p_{10}, p_{20}, \ldots, p_{\ell 0}),$$
$$\overline{p_x} = (p_{1k1}, p_{1k2}, \ldots, p_{1km}, \ldots, p_{\ell k1}, p_{\ell k2}, \ldots, p_{\ell km}),$$
$$k = \overline{1,r+1}\ .$$

From the above assumptions, the stationary distribution p_x, $x \in X$, is the unique solution of the steady state equilibrium equations

$$\overline{p}_0^T \Lambda + \overline{p}_1^T (I \otimes \overline{\mu}) = \overline{0}^T, \tag{7}$$

$$\overline{p}_0^T (\overline{\lambda}\ \overline{\alpha}^T \otimes \overline{\beta}^T) + \overline{p}_1^T(\Lambda \otimes I + I \otimes M) + \overline{p}_2^T (I \otimes \overline{\mu}\ \overline{\beta}^T) = \overline{0}^T, \tag{8}$$

$$\overline{p}_{k-1}^T (\overline{\lambda}\ \overline{\alpha}^T \otimes I) + \overline{p}_k^T(\Lambda \otimes I + I \otimes M) + \overline{p}_{k+1}^T (I \otimes \overline{\mu}\ \overline{\beta}^T) = \overline{0}^T, \quad k = \overline{2,r}\ ; \tag{9}$$

$$\overline{p}_r^T (\overline{\lambda}\ \overline{\alpha}^T \otimes I) + \overline{p}_{r+1}^T((\Lambda + \overline{\lambda}\ \overline{\alpha}^T) \otimes I + I \otimes M) = \overline{0}^T, \tag{10}$$

where

$$\overline{\lambda} = -\Lambda \overline{1}\ ,\ \mu = -M\overline{1},$$

with the normalizing condition

$$\sum_{k=0}^{r+1} \overline{p}_k^T \overline{1} = 1. \tag{11}$$

Let us defined matrices

$$\tilde{\Lambda} = (\bar{1}\,\bar{\alpha}^T - I) \otimes M - \Lambda \otimes I, \quad \tilde{M} = \Lambda \otimes (\bar{1}\,\bar{\beta}^T - I) - I \otimes M.$$

We shall now study some properties of the stationary distribution p_x, $x \in X$, based on results in [3].

1°) Multiplying on the right both sides of equations (8)-(10) by the matrix $I \times \bar{1}$, yields

$$\bar{p}_o^T\,\bar{\lambda}\,\bar{\alpha}^T + \bar{p}_1^T\,(\Lambda \otimes \bar{1} - I \otimes \bar{\mu}) + \bar{p}_2^T(I \otimes \bar{\mu}) = \bar{0}^T, \tag{12}$$

$$\bar{p}_{k-1}^T(\bar{\lambda}\,\bar{\alpha}^T \otimes \bar{1}) + \bar{p}_k^T(\Lambda \otimes \bar{1} - I \otimes \bar{\mu}) + \bar{p}_{k+1}^T(I \otimes \bar{\mu}) = \bar{0}^T, \tag{13}$$

$$k = \overline{1,r},$$

$$\bar{p}_r^T(\bar{\lambda}\,\bar{\alpha}^T \otimes \bar{1}) + \bar{p}_{r+1}^T((\Lambda \oplus \bar{\lambda}\,\bar{\alpha}^T) \otimes \bar{1}) - I \otimes \bar{\mu}) = \bar{0}^T. \tag{14}$$

Sum (13) with respect to k and combine the result with (7), (12) and (14), to get, after modest rearrangement,

$$[\bar{p}_o^T + \sum_{k=1}^{r+1} \bar{p}_k^T(I \otimes \bar{1})](\Lambda + \bar{\lambda}\,\bar{\alpha}^T) = \bar{0}^T. \tag{15}$$

2°) After multiplying on the right both sides of (7), (12) and (13) by the vector $\bar{1}$, we obtain

$$\bar{p}_o^T\,\bar{\lambda} - \bar{p}_1^T\,(1 \otimes \bar{\mu}) = 0,$$

$$\bar{p}_o^T\,\bar{\lambda} - \bar{p}_1^T(\bar{\lambda} \otimes \bar{1} + \bar{1} \otimes \bar{\mu}) + \bar{p}_2^T(\bar{1} \otimes \bar{\mu}) = 0,$$

$$\bar{p}_{k-1}^T(\bar{\lambda} \otimes \bar{1}) - \bar{p}_k^T(\bar{\lambda} \otimes \bar{1} + \bar{1} \otimes \bar{\mu}) + \bar{p}_{k+1}^T(\bar{1} \otimes \bar{\mu}) = 0,$$

$$k = \overline{2,r}.$$

The latter equalities together with (7) and (16) give

$$\bar{p}_1^T \tilde{M} = \bar{p}_1^T(I \otimes \bar{\mu}\, \bar{\beta}^T) = -\bar{p}_o^T(\Lambda \otimes \bar{\beta}^T), \tag{17}$$

$$\bar{p}_k^T \tilde{M} = \bar{p}_k^T(I \otimes \bar{\mu}\, \bar{\beta}^T) + \bar{p}_{k-1}^T(\bar{\lambda}\, \bar{\alpha}^T \otimes I) - \bar{p}_{k-1}^T(\bar{\lambda}\, \bar{\alpha}^T \otimes \bar{1}\, \bar{\beta}^T) =$$

$$= \bar{p}_k^T(I \otimes \bar{\mu}\, \bar{\beta}^T) + \bar{p}_{k-1}^T(\bar{\lambda}\, \bar{\alpha}^T \otimes I) - \bar{p}_k^T(\bar{1}\, \bar{\alpha}^T \otimes \bar{\mu}\, \bar{\beta}^T), \quad k = \overline{2,r}. \tag{18}$$

4°) After multiplying at the right both sides of (8) and (9) by the matrix $(\bar{1}\, \bar{\alpha}^T - I) \otimes I$, we have

$$\bar{p}_k^T(\Lambda \otimes I + I \otimes M)[(\bar{1}\, \bar{\alpha}^T - I \otimes I] + \bar{p}_{k+1}^T[(\bar{1}\, \bar{\alpha}^T - I) \otimes \bar{\mu}\, \bar{\beta}^T] = \bar{0}^T, \quad k = \overline{1,r}.$$

This implies that

$$\bar{p}_k^T \tilde{\Lambda} = \bar{p}_k^T(\bar{\lambda}\, \bar{\alpha}^T \otimes I) + \bar{p}_{k+1}^T[(I - \bar{1}\, \bar{\alpha}^T) \otimes \bar{\mu}\, \bar{\beta}^T], \; k=\overline{1,r}. \tag{19}$$

5°) Now, from (17)-(19) and (10), it follows that :

$$\bar{p}_1^T \tilde{M} = -\bar{p}_o^T(\Lambda \otimes \bar{\beta}^T), \tag{20}$$

$$\bar{p}_k^T \tilde{M} = \bar{p}_{k-1}^T \tilde{\Lambda}, \quad k = \overline{2,r}, \tag{21}$$

$$\bar{p}_{r+1}^T [\Lambda + \bar{\lambda}\, \bar{\alpha}^T) \otimes I + I \otimes M] - \bar{p}_r^T(\bar{\lambda}\, \bar{\alpha}^T \otimes I). \tag{22}$$

We note that the substable and irreducible matrix $\Lambda + \bar{\lambda}\, \bar{\alpha}^T$ is the generator of an homogeneous Markov process. The eigenvalues of $(\Lambda + \bar{\lambda}\, \bar{\alpha}^T) \otimes I + I \otimes M$ are the sum of the eigenvalues of the substable matrix $\Lambda + \bar{\lambda}\, \bar{\alpha}^T$ and of the stable matrix M [11]. Hence $(\Lambda + \bar{\lambda}\, \bar{\alpha}^T) \otimes I + I \otimes M$ is stable and invertible. Lemma 2) entails that the matrices $\tilde{\Lambda}$ and $\tilde{M}$ are also invertible.

Introduce

$$W_0 = -(\Lambda \otimes \bar{\beta}^T)\,\tilde{M}^{-1}, \quad W = \tilde{\Lambda}\,\tilde{M}^{-1},$$

$$W_r = -(\bar{\lambda}\,\bar{\alpha}^T \otimes I)\,[(\Lambda + \bar{\lambda}\,\bar{\alpha}^T) \otimes I + I \otimes M]^{-1},$$

$$V = I + \Big(\sum_{k=0}^{r-1} W_0 W^k + W_0 W^{r-1} W_r\Big)(I \otimes \bar{1}).$$

<u>Theorem</u>. Let $\bar{u}$ be a solution of the steady-state system of equations

$$\begin{aligned} &\bar{u}^T(\Lambda + \bar{\lambda}\,\bar{\alpha}^T) = \bar{0}^T \\ &\bar{u}^T\,\bar{1} = 1. \end{aligned} \tag{23}$$

Then, the stationary distribution p_x, $x \in X$, has the form

$$\bar{p}_k^T = \begin{cases} \bar{p}_0^T\, W_0 W^{k-1}, & k = \overline{1,r}, \\ \bar{p}_0^T\, W_0 W^{r-1}\, W_r, & k = r+1, \end{cases} \tag{24}$$

and the vector $\bar{p}_0$ satisfies the system

$$\bar{p}_0^T\, V = \bar{u}^T. \tag{25}$$

<u>Proof</u>. The irreducibility of the matrix $\Lambda + \bar{\lambda}\,\bar{\alpha}^T$ implies that the system (23) has a unique solution. From the relations (20)-(22), we find the stationary distribution p_x, $x \in X$, in the form (24). Moreover (24) and (15) yield that the vector $\bar{p}_0^T V$ satisfy the system

$$\bar{p}_0^T V(\Lambda + \bar{\lambda}\,\bar{\alpha}^T) = \bar{0}^T.$$

Therefore $\bar{p}_0^T V = c\,\bar{u}^T$, where c is a constant. The normalizing condition (11) and relation (24) lead to c = 1. The proof is concluded. □

4 - A NUMERICAL EXAMPLE

In order to illustrate the theoretical derivarions in this paper, some numerical results for the QSs $H_3M/E_4/1/10$, $E_3/H_4MM/1/10$ and $E_3/E_4/1/10$ are presented. The Erlang and hyperexponential distributions are obvious by particular PH-distributions. In the case of the QS $H_3M/E_4/1/10$ e.g. we have respectively

$$G = \mathrm{diag}(-\lambda_1, -\lambda_2, -\lambda_3) \;, \quad \bar{\alpha}^T = (\alpha_1, \alpha_2, \alpha_3) \;,$$

$$\lambda_i > 0, \; \alpha_i > 0, \; i = \overline{1,3}, \; \sum_{i=1}^{3} a_i = 1 \text{ and}$$

$$G = \begin{vmatrix} -\mu & \mu & 0 & 0 \\ 0 & -\mu & \mu & 0 \\ 0 & 0 & -\mu & \mu \\ 0 & 0 & 0 & -\mu \end{vmatrix} \;, \quad \bar{\beta}^T = (1,0,0,0),$$

with $\mu > 0$.

The calculation have been carried out at the department of mathematics of the University of Friendship Patrice Lumumba. The program was written by S.S. Spesivov in Fortran language.

The distribution parameters were given following numerical values :

<u>H_3M/E_4</u> : $\bar{\lambda}^T = (1.0,\ 2.0,\ 3.0)$, $\bar{\alpha}^T = (0.2,\ 0.3,\ 0.5)$; $\mu = 12.0$;

<u>E_3/H_4M</u> : $\lambda = 10.5$; $\bar{\mu}^T = (1.0,\ 2.0,\ 3.0,\ 4.0)$, $\bar{\beta}^T = (0.1,\ 0.2,\ 0.3,\ 0.4)$;

<u>E_3/E_4</u> : $\lambda = 7.5$; $\mu = 14.0$

[λ and μ are respectively the mean arrival rate and the mean service rate.].

In Table 1 the values $p_k \overset{def}{=} \sum_i \sum_j p(i,k,j)$, $k = \overline{0,11}$, are given.

The last row of Table 1 check the normalizing condition (11).

Table 1

QS / k	$H_3M/E_4/1/10$	$E_3/H_4M/1/10$	$E_3/E_4/1/10$
0	0.355551	0.003984	0.285716
1	0.248652	0.007517	0.431506
2	0.162687	0.010527	0.189542
3	0.097829	0.015112	0.063441
4	0.057377	0.021910	0.020310
5	0.033436	0.031891	0.006469
6	0.019456	0.046486	0.002059
7	0.011321	0.067787	0.000658
8	0.006597	0.098770	0.000209
9	0.003856	0.143269	0.000066
10	0.002279	0.204365	0.000021
11	0.000962	0.348383	0.000005
$\sum$	1.000003	1.000001	1.000002

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